Department of
Industrial Engineering & Management

BGU Engineering
Research. Connect. Impact

Scheduling Intervals

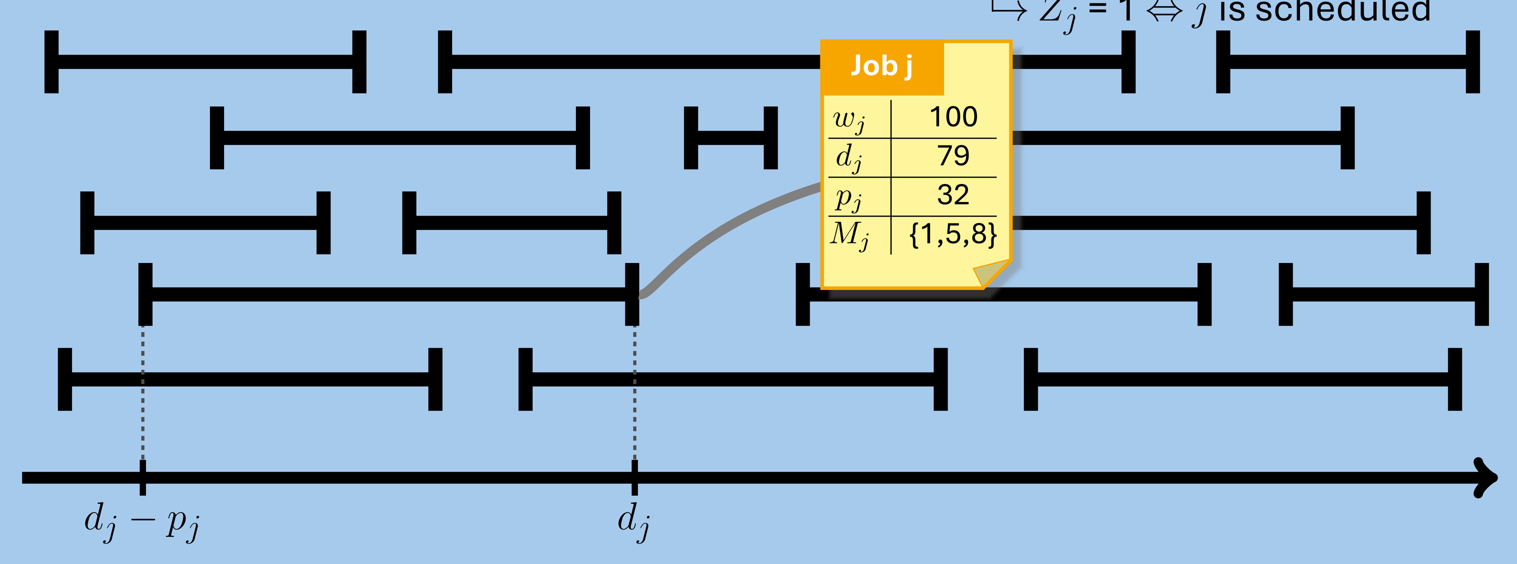
Student: Yuval Itzhaki
Advisors: Prof. Danny Hermelin & Prof. Dvir Shabtay

Interval Scheduling with Eligible Machines

Input: A set of n jobs $\mathcal{J} = \{1, \ldots, n\}$, a set of m machines, and an integer W^*. Each job $j \in \mathcal{J}$ has a processing time $p_j \in \mathbb{N}$, a due date $d_j \in \mathbb{N}$, a weight $w_j \in \mathbb{N}$, and a set of eligible machines $M_j \subseteq \{1, \ldots, m\}$.

Question: Is there a feasible schedule for $\mathcal{J}$ such that $\Sigma w_j Z_j \geq W^*$?

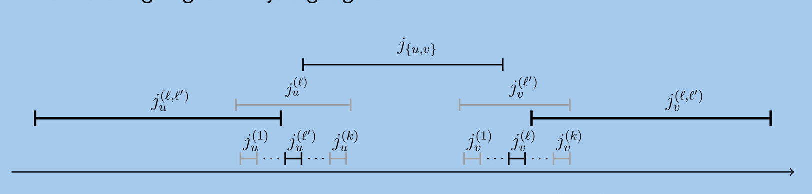
↳ $Z_j = 1 \Leftrightarrow j$ is scheduled



Our Contribution

Theorem 1. Interval Scheduling with Eligible Machines is strongly W[1]-hard w.r.t. m.

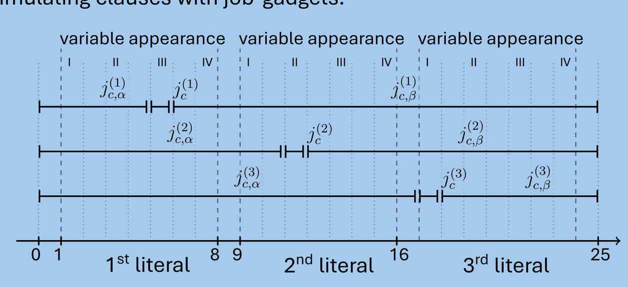
- Reduction from Multicolored Clique: Simulating edges with job-gadgets.



The edge-selection gadget of the reduction for the edge $\{u, v\}$ of the original graph.

Theorem 2. Interval Scheduling with Eligible Machines NP-hard for constant $p_{\max}$.

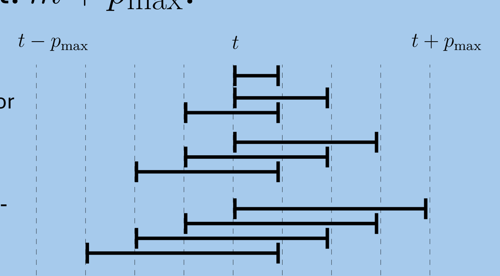
- Reduction from Exact (3-4)-SAT: Simulating clauses with job-gadgets.



The reduction applied to the clause $(x \vee y \vee \neg z)$.

Theorem 3. Interval Scheduling with Eligible Machines is FPT w.r.t. $m + p_{\max}$.

- Data reduction: Deleting irrelevant jobs. Retaining top-weighted jobs for each timestep t.
- $O(m^2 p_{\max}^2)$ jobs remain per t.
- Dynamic program on the time-steps solves the problem in $O((m^2 p_{\max}^2)^{2m} nm)$-time.



Visualization of the relevant jobs for time-step t.

Related Work

Arkin & Silverberg

Scheduling Jobs with Fixed Start and End Times — 1987

- Proved NP-hardness.
- Gave an $O(n^{m+1})$-time algorithm

Mnich & van Bevern

Parameterized Complexity of Machine Scheduling: 15 Open Problems — 2018

- Posed **Open Question 8:**

  Is the Problem FPT w.r.t. m?

Hermelin, Itzhaki, Molter, Shabtay

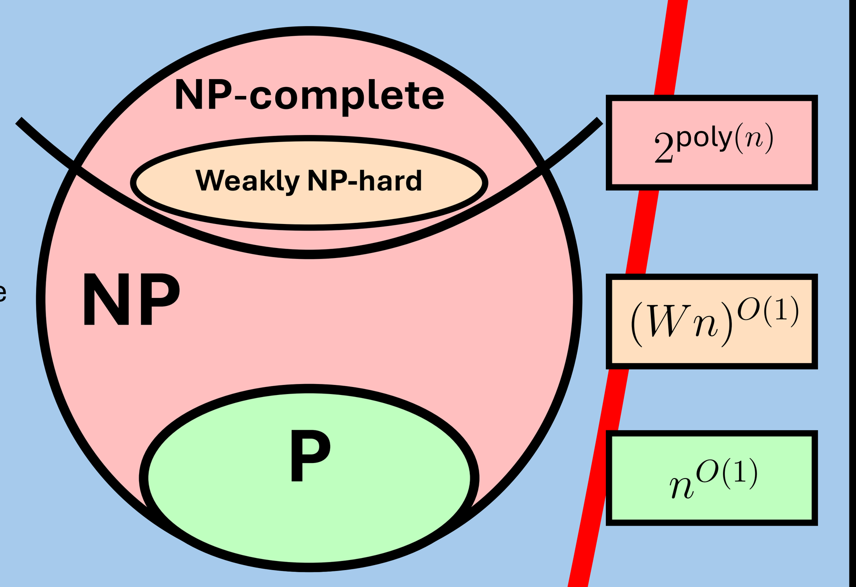
On the Parameterized Complexity of Interval Scheduling with Eligible Machine Sets — 2024

- Answered Open Question 8 negatively.
- Left open whether an $(n^{O(\sqrt{m})})$-time algorithm exists.

Classical Complexity

Studies how running time depends on input size n:

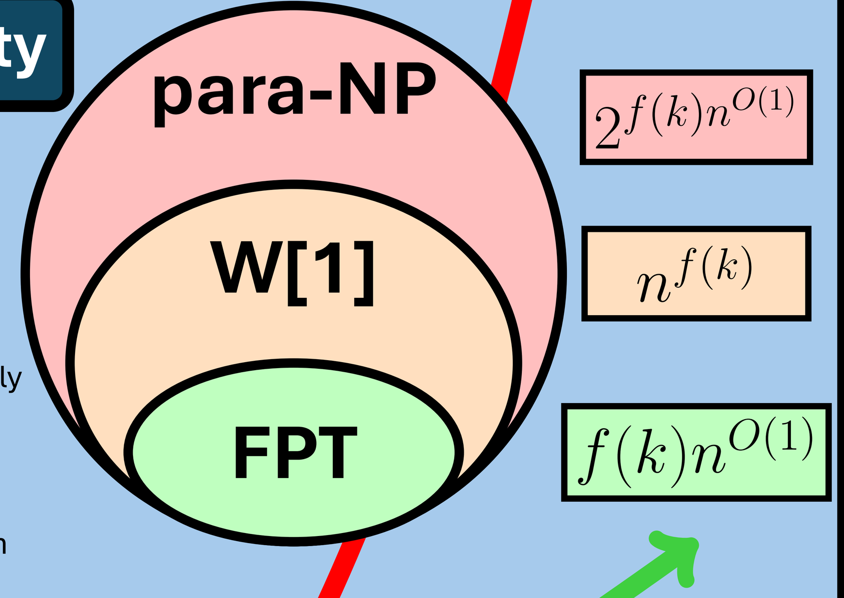
- Class P contains problems solvable in time
$$n^{O(1)}.$$
- A problem is NP-hard if every problem in NP can be reduced to it. Such problems are solvable in exponential time
$$2^{\text{poly}(n)}$$
and are believed not to admit polynomial-time algorithms.
- A problem is weakly NP-hard if its hardness arises from large numerical values in the input. Such problems admit pseudo-polynomial-time algorithms, whose runtime is
$$(Wn)^{O(1)}$$
by which W is the largest numeric input value.



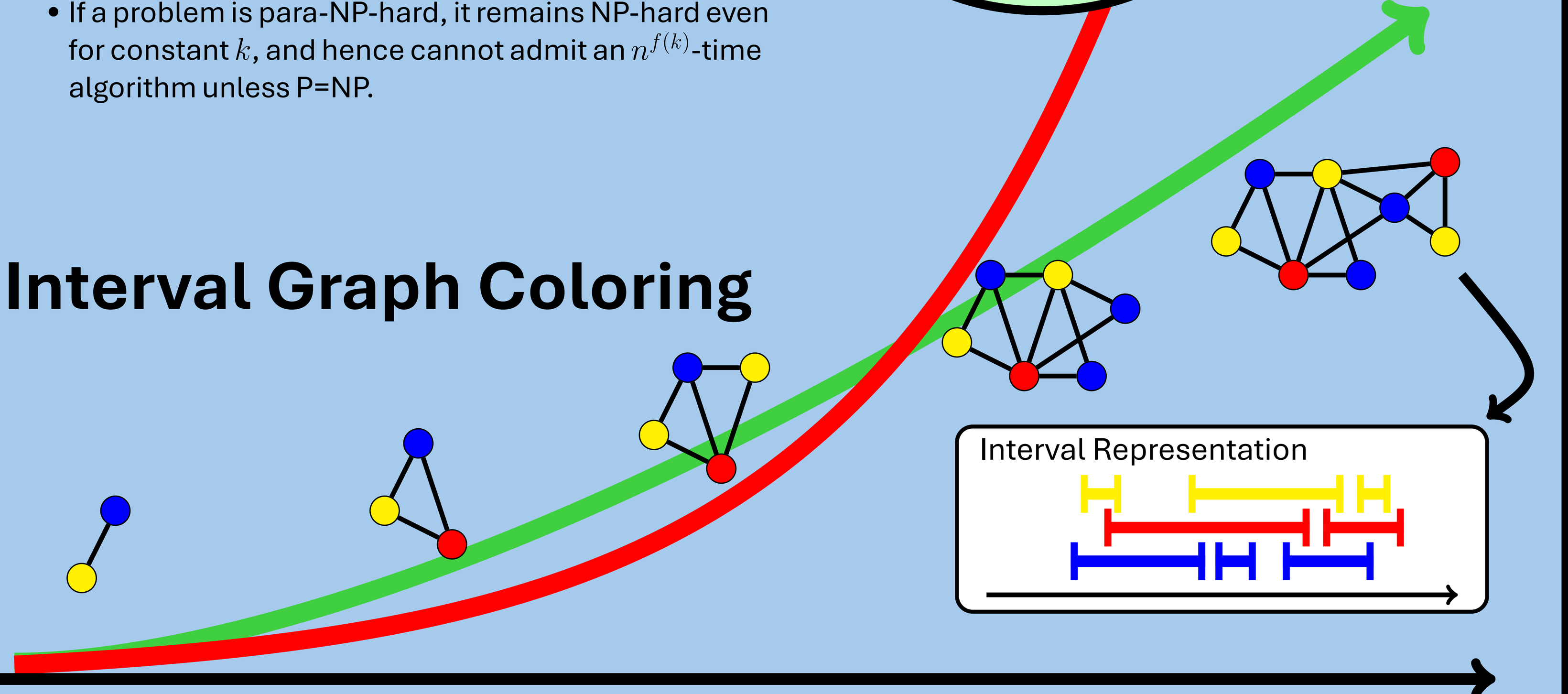
Parameterized Complexity

Studies the effect of input parameters on running time:

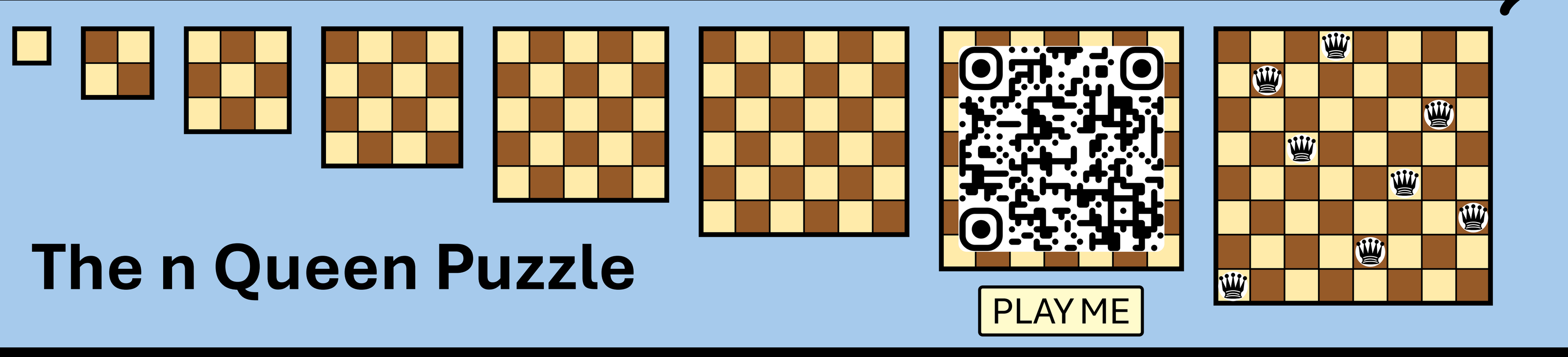
- A problem is fixed-parameter tractable (FPT) with respect to a parameter k if it can be solved in time
$$f(k) \cdot n^{O(1)}.$$
- If a problem is W[1]-hard with respect to k, it is unlikely to admit an FPT algorithm, the best we typically hope for is running time
$$n^{f(k)}.$$
- If a problem is para-NP-hard, it remains NP-hard even for constant k, and hence cannot admit an $n^{f(k)}$-time algorithm unless P=NP.



Interval Graph Coloring

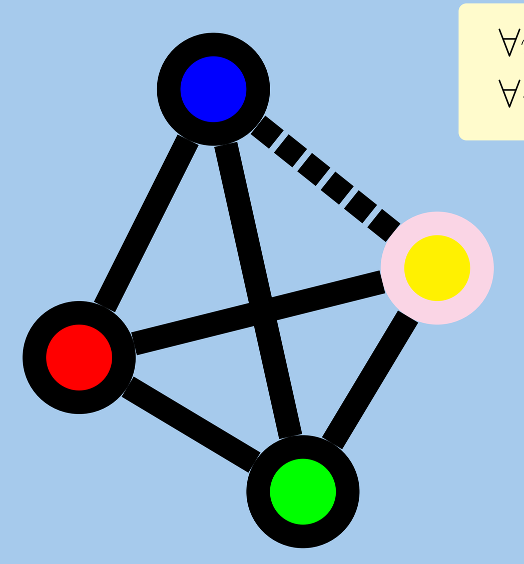


The n Queen Puzzle



Theorem 1. W[1]-hardness

Multicolored Clique



$\forall v \in V$ create VERTEX JOBS
$\forall \{u, v\} \in E$ create an EDGE JOB

Input: A graph G with a vertex-coloring and an integer k.
Question: Is there a k-sized clique with valid coloring?

Multicolored Clique is W[1]-hard w.r.t. the clique size k [Fellows, Hermelin, Rosamond, Viallette 2009].

Interval Scheduling with Eligible Machines

#machines
$$\binom{k}{2} + 1$$